THE HYDRODYNAMICS OF A FERROMAGNETIC FLUID

V. M. Zaitsev and M. I. Shliomis

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§1. Recently, in connection with various engineering applications, interest has arisen in the hydrodynamics of a ferromagnetic fluid. Such a medium can be realized by colloidal dispersion of fine ferromagnetic particles in an ordinary fluid. The magnetic moment of unit volume **M** can reach considerable values, becoming comparable with the magnetic moment of solid ferromagnetics.

Equations for a ferromagnetic fluid were first derived in [1]:

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\,\mathbf{v}\right]=-\nabla p+\eta\nabla^2\mathbf{v}+M\nabla H, \qquad (1.1)$$

$$\frac{\partial T}{\partial t}+\mathbf{v}\nabla T+\frac{T}{c_H}\left(\frac{\partial \mathbf{M}}{\partial T}\right)_H\left[\frac{\partial \mathbf{H}}{\partial t}+(\mathbf{v}\nabla)\,\mathbf{H}\right]=\chi\nabla^2 T, \qquad (1.2)$$

$$\operatorname{div}\mathbf{v}=0, \quad \operatorname{div}(\mathbf{H}+4\pi\mathbf{M})=0, \quad \operatorname{rot}\mathbf{H}=0. \qquad (1.3)$$

In the derivation of these equations it was assumed that the fluid was magnetized to saturation by a strong magnetic field. Therefore, the absolute value of the magnetization M is not a function of the field H and is determined only by temperature. If the magnetic fields are not too strong and saturation is not attained, in Eq. (1.1) we must have $(\mathbf{M}\nabla)\mathbf{H}$ instead of $M\nabla H$.

System (1.1–1.3) is considerably more complicated than the usual equations of hydrodynamics. This is due, in the first place, to the considerable nonlinearity of the magnetic force $M\nabla H$ in the Navier-Stokes equation. Furthermore, motion in an nonuniform magnetic field is always nonisothermal, and this is due to cooling of the magnetic when it moves into weak-field regions. Consequently, in the cases of the nonuniform magnetic fields that are of practical interest, the well-known one-dimensional solutions of ordinary hydrodynamics (Poiseuille flow, Couette flow, etc.) cease to be one-dimensional and, as a result, are inaccurate. In §§2 and 3, we consider ferrohydrodynamic analogies of Poiseuille and Couette flows.

§2. Let us consider the motion of a viscous ferromagnetic fluid in a plane layer of thickness $2l$, caused by nonuniformity of the magnetic field. A strong uniform field H_0 is directed across the layer; a field gradient along the layer is necessary for motion of the fluid. Then, the occurrence of a longitudinal magnetic-field component that varies across the layer follows from Eq. (1.3.3). In a nonmagnetized medium ($M = 0$), Eqs. (1.3.2) and (1.3.3) are satisfied by

$$H_z=H_0+Ax, \qquad H_x=Az. \qquad (2.1)$$

The presence of **M** in (1.3.2) causes solution (2.1) to no longer satisfy Maxwell equations (1.3.2) and (1.3.3). We limit ourselves to low field gradient

$$Al\ll H_0 \qquad (2.2)$$

As will be seen below, even under this condition the magnetic forces are equivalent to considerable pressure differentials. It should also be borne in mind that when condition (2.2) is not satisfied, the field drops to zero at distances comparable with the layer thickness. This violates the fundamental assumption of total magnetization of the medium. When condition (2.2) is satisfied, solution (2.1) for the field and

$$M_z=M_0, \qquad M_x=(M_0\,/\,H_0)\,Az \qquad (2.3)$$

for the moment satisfy Eqs. (1.3.2) and (1.3.3) in an approximation linear in A. Here M_0 is the saturation magnetization, which is a function only of temperature, and in a not too wide interval of variation it can be represented by the formula [1]

$$M_0=a(\theta-T). \qquad (2.4)$$

Here, a and θ are positive constants and T is the absolute temperature. Substituting the **M** and **H** found into Eq. (1.1), we obtain

$$v_x=(2\eta)^{-1}A\,M_0\,(l^2-z^2), \qquad v_z=0. \qquad (2.5)$$

If we compare this motion with ordinary Poiseuille flow, we see that the role of the pressure differential Δp is played here by $M_0\Delta H$. Let us estimate the effect. if the volume concentration of ferromagnetic particles is on the order of 0.1, then $M_0\sim 10^2$ erg/G · cm^3, and when $\Delta H\sim 10^4$ Oe we obtain an effective $\Delta p\sim 1$ atm. This magnetic pressure can equalize the hydrodynamic pressure differential (magnetic mirror).

In an approximation quadratic in Al/H_0, the longitudinal velocity does not vary, and from (1.2) we find the temperature distribution over the layer thickness

$$T=T_0\left[1+\frac{A^2M_0a}{24\chi\eta c_H}(5l^4-6l^2z^2+z^4)\right]. \qquad (2.6)$$

Temperature inhomogeneity is caused by the magnetocaloric effect, i.e., by heating of the magnet when it moves into the strong-field region. Since the most intense fluid motion occurs near the center of the layer, the maximum temperature is also found there. This, in turn, results in partial demagnetization of the heated regions of the fluid (2.4), causing additional magnetic forces directed across the layer. Together with inhomogeneity of H_x, this results, in an approximation quadratic in A, in the appearance of a transverse velocity component (magnetic convection), i.e., the motion ceases to be one-dimensional.

§3. Let us consider a ferrohydrodynamic analog of Couette flow. Let there be a plane layer with boundaries $z=\pm l$ that move along the x-axis with the velocities $\pm V$. The same assumptions as in the previous case are made about the magnetic field, i.e., condition (2.2) is assumed to be satisfied. Here, however, formulas (2.1) and (2.3) are not valid even in an approximation linear in A.

This follows from Eq. (1.2), in which for v we can substitute the velocity of unperturbed Couette flow Vz/l

$$V\frac{z}{l}\left(\frac{\partial T}{\partial x}-\frac{aT_0A}{c_H}\right)=\chi\nabla^2T. \qquad (3.1)$$

The solution of this equation has the form

$$T = T(z) = T_0\left[1 + \frac{AVa}{6\chi l c_н} z(l^2 - z^2)\right]. \qquad (3.2)$$

By virtue of (2.4) and (3.2), magnetization is now already a function of z in an approximation that is linear in the field gradient. If we solve (1.3.2) and (1.3.3) with the M(z) found, we obtain

$$H_z = H_0 + Ax + \frac{4\pi a^2 T_0 VA}{6\chi l c_н} z(l^2 - z^2) + O(A^2),$$

$$H_x = A_z + O(A^2),$$

$$M_z = a(\theta - T_0) - \frac{a^2 T_0 VA}{6\chi l c_н} z(l^2 - z^2) + O(A^2),$$

$$M_x = aAH_0^{-1}(\theta - T_0)z + O(A^2), \qquad (3.3)$$

Now we determine the corrections for velocity and pressure. We seek v in the form

$$v_x = Vz/l + w(z), \qquad v_z = 0, \qquad (3.4)$$

where the first term is conventional Couette flow and w(z) is a correction governed by the magnetic forces. Then from (1.1) we obtain

$$0 = -\frac{\partial p}{\partial x} + \eta w'' + aA(\theta - T_0),$$

$$0 = -\frac{\partial p}{\partial z} + \frac{4\pi a^2 T_0 VA}{6\chi l c_н}(l^2 - 3z^2). \qquad (3.5)$$

The former of these equations determines w(z) and the latter gives the pressure distribution over the layer cross section. It is easy to see that, in the absence of a hydrodynamic pressure gradient along the layer, Eqs. (3.5) lead to the "Poiseuille" correction for the velocity of Couette motion

$$w(z) = (2\eta)^{-1} AM_0 (l^2 - z^2). \qquad (3.6)$$

The interaction of Poiseuille and Couette flows, which is governed by nonlinearity of the magnetic force, can be detected only in approximations higher in A.

§4. It was shown above that a magnetic-field gradient can cause motion of the medium. Because of this, it is advisable to find out in which cases equilibrium of the fluid in the presence of ∇H is possible. Application of the rot operation to (1.1) gives it the form

$$\frac{\partial}{dt}\operatorname{rot}\mathbf{v} =$$

$$= \operatorname{rot}(\mathbf{v}\times\operatorname{rot}\mathbf{v}) + \nu\nabla^2\operatorname{rot}\mathbf{v} + \frac{1}{\rho}\nabla M \times \nabla H. \qquad (4.1)$$

Hence, we see that a necessary (but not sufficient) condition for fluid equilibrium is $\nabla M \times \nabla H = 0$, or, in view of (2.4),

$$\nabla T \times \nabla H = 0. \qquad (4.2)$$

Thus, in the presence of ∇H, equilibrium is possible either if T = const and the field gradient is equalized by external pressure or if $\nabla T \parallel \nabla H$. In the latter case, however, the problem of stability of the possible equilibrium arises. In the example considered below, the problem of equilibrium stability of a nonuniformly heated fluid in a magnetic field has a simple solution.

Let the fluid occupy a thin cylindrical layer with radii R and R + δ, where δ ≪ R. A constant electric current I flows over the inside cylinder, and creates in the layer the magnetic field

$$H_\varphi = H = 2I/cr. \qquad (4.3)$$

The temperature at the layer boundaries is given

$$T(R) = T_1, \qquad T(R+\delta) = T_1 - \vartheta \qquad (\vartheta \ll T_1). \qquad (4.4)$$

It is easy to see that Eqs. (1.1–1.3) are satisfied by the equilibrium solution

$$\mathbf{v} = 0, \ T_0 = T_1 - \vartheta x/\delta, \ H_\varphi = H,$$

$$M_\varphi = M_0 = a(\theta - T_1 + \vartheta x/\delta)\ (x = r - R). \qquad (4.5)$$

Let us investigate the stability of this equilibrium with respect to axisymmetric perturbations that are periodic along the z-axis. For this, we substitute into Eqs. (1.1–1.3):

$$\mathbf{v},\ p = p_0 + p',\ T = T_0 + T',\ M_\varphi = M_0 + m, \qquad (4.6)$$

where v, p', T', and m are the perturbations of the corresponding quantities. If we linearize the equations in the small perturbations and assume that all derivatives with respect to time are zero (stability boundary), we find

$$0 = -\nabla p' + \eta\nabla^2\mathbf{v} + m\nabla H, \qquad \operatorname{div}\mathbf{v} = 0,$$

$$\mathbf{v}\nabla T_0 + \frac{T_0}{c_н}\left(\frac{\partial M_0}{\partial T}\right)_н(\mathbf{v}\nabla)H = \chi\nabla^2 T'. \qquad (4.7)$$

Equation (1.3.2) is satisfied identically for axisymmetric perturbations. We seek the solution of system (4.4) as

$$v_z = w(r)\cos kz,\ \ v_r = v(r)\sin kz,\ \ v_\varphi = 0,\ \ p' = s(r)\sin kz,$$

$$T' = \tau(r)\sin kz,\ \ m = -aT'. \qquad (4.8)$$

Since the layer is thin (δ ≪ R), we can make a number of simplifications. First, the sole (radial) component of ∇H can be written as

$$(\nabla H)_r = -2I/cr^2 \approx -HR^{-1}.$$

Furthermore, from the equation div**v** = 0 it follows that

$$v' = kw. \qquad (4.9)$$

The remaining equations of system (4.7) give

$$s' = \eta(v'' - k^2 v) + a\tau\frac{H}{R}, \qquad ks = \eta(w'' - k^2 w),$$

$$-\frac{\vartheta}{\delta}v + \frac{aT_1 H}{Rc_н}v = \chi(\tau'' - k^2\tau). \qquad (4.10)$$

In the latter equation, T_0 is replaced by T_1, with use of the smallness of ϑ as compared with T_1. System (4.9–4.10) is to be solved under the following boundary conditions when x equals 0 and δ:

$$v = w = \tau = 0. \qquad (4.11)$$

Eliminating s, v, and w from the equations, we obtain

$$\left(\frac{d^2}{dx^2} - k^2\right)^3\tau + Ck^2\tau = 0, \quad C = \frac{aH}{\varkappa\eta R}\left(\frac{\vartheta}{\delta} - \frac{aT_1 H}{Rc_н}\right). \qquad (4.12)$$

The solution of boundary-value problem (4.11–4.12) (in the first condition for v and w we can substitute the conditions for the higher derivatives of τ) determines $C(k^2)$, and the minimum of this function corresponds to the stability boundary. It should be noted that the problem in question is entirely equivalent after the approximations to the problem of stability of a plane horizontal fluid layer heated from below in a gravity field, wherein $C\delta^4$ plays the role of the Rayleigh number. Therefore, we can use the calculation results of Pellew and Southwell [2], and this yields $(C\delta^4)_{min} = 1710$, or for the critical temperature gradient:

$$A = \frac{\vartheta_{min}}{\delta} = 1710\frac{\chi\eta R}{aH\delta^4} + \frac{aT_1 H}{Rc_н} \qquad \left(A = 1710\frac{\chi\eta}{\rho g\beta\delta^4}\right). \qquad (4.13)$$

As should be expected, instability occurs only when ϑ is positive.

The expression for the critical temperature gradient in the problem of convective stability of a plane layer [2] is given in (4.13) for comparison.

As is apparent, the magnetic field gradient HR^{-2} plays a dual role. On the one hand, it is equivalent to the gravity field g, i.e., it causes convection; the pyromagnetic coefficient α is equivalent in this case to the coefficient of volume expansion β. On the other hand, the magnetic field is a stabilizing factor (the second term in (4.13)). This is explained by cooling of the fluid when it moves into the weak-field region. Therefore, the critical temperature gradient as a function of HR^{-1} has a minimum when

$$\left(\frac{H}{R}\right)_0^2 = 1710\frac{\chi\eta c_H}{\alpha^2 T_1 \delta^4}. \tag{4.14}$$

This field gradient corresponds to

$$A_0 = \frac{2\alpha T_1}{c_H}\left(\frac{H}{R}\right)_0.$$

An estimate according to (4.14) gives for $(HR^{-1})_0$ a value on the order of $\delta^{-2}(10^3-10^4)$ Oe/cm. Thus, the magnetic field has a stabilizing influence only for large values of its gradient. In all situations of practical interest, therefore, the critical temperature gradient decreases as the field increases.

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